the propagation of stress and deformation waves in pipes one must take into account the quasi-two-dimensional nature of the deformed state of the walls.

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GENERATION OF INTERNAL WAVES UNDER THE COMBINED TRANSLATIONAL AND VIBRATIONAL MOTION OF A CYLINDER IN A FLUID BILAYER

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The analysis of internal waves in an inviscid fluid bilayer has been considered in the linear theory for a general form of the motion of the source (see, e.g., [1]). For the special case of the motion of a circular cylinder perpendicular to its generatrix, one of the interesting regimes occurs when the cylinder, translating parallel to the surface, simultaneously performs vertical harmonic oscillations. As shown in [1], the wave field in this case depends in an essential way on the oscillation frequency Ω. For relatively small frequencies waves are excited both in front of and behind the body. When the frequency increases above a certain critical value Ω_* (which depends on the translational velocity of the body, the thicknesses of the fluid layers, and the density difference between them) wave motion is only possible behind the body. When $\Omega = \Omega_*$, the linear theory of an ideal fluid predicts an unbounded growth (as a power law) of the wave amplitude in time, as occurs in resonance phenomena of various kinds. The growth of the wave can be bounded either by viscosity or by nonlinear effects. The effect of viscosity was considered in [2] for a similar plane problem involving excitations created by a horizontally oscillating cylinder moving in a lower layer of an infinite fluid bilayer. In the problem considered in [2], it was assumed that the viscosity was nonzero only in the upper layer. Nonlinear effects have been analyzed in [3, 4], where for the special case of a uniform fluid, nonlinear boundary conditions on the free surface were taken into account. The behavior of internal waves in a linearly stratified fluid has been studied theoretically and experimentally for various types of the motion of the body (see, for example, [5]). The formulation of the problem closest to the one considered here is that of [6].

In the theoretical part of the present paper we are concerned mainly with taking into account viscosity in the framework of the linear model. We also performed experiments in which the critical and near-critical regimes were studied. The present paper is a continuation of [7], where theoretical and experimental results were presented for internal waves generated by the vertical harmonic oscillations of a submerged cylinder in a bilinear of viscous fluid with surface tension at the boundary.

In the theoretical solution of the linear problem for the behavior of internal waves generated by a moving circular cylinder, the cylinder is modelled as a point dipole. The fluid is assumed to be incompressible, is at rest in the unperturbed state, and consists of two infinitely deep layers with small viscosities; the density of the fluid in the upper ($y > 0$) and lower ($y < 0$) layers are ρ_1 and $\rho_2 = \rho_1(1 + \varepsilon)(\varepsilon > 0)$, respectively, and the dynamical coefficients of viscosity are μ_1 and μ_2. The surface tension on the boundary

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between the layers is σ. The y axis is directed vertically upward, the horizontal x axis lies in the unperturbed surface. It is assumed that at t = 0 in the fluid in the upper layer a dipole of variable dipole moment $\mathbf{M}(t)$ ($\mathbf{M}(t) \equiv 0$ and $t \le 0$) begins to act and the trajectory of the dipole has the form X = Y(t), where $\mathbf{X} = (x, y)$, $\mathbf{Y}(t) = (y_1(t), y_2(t))$. In a uniform infinite fluid, this is equivalent to the motion of a circular cylinder of radius R with velocity $\mathbf{U}(t) = (U_1(t), U_2(t))$ (the dipole moment is equal to $\mathbf{M}(t) = 2\pi R^2\mathbf{U}(t)$, $d\mathbf{Y}/dt = \mathbf{U}(t)$). Finally an expression for the function η(x, t), describing the vertical displacement of the surface between the two layers induced by the motion of the cylinder in the upper layer can be written as (for a detailed discussion, see [7])

$$\eta = -\frac{2R^2}{2+\varepsilon}\int_0^\infty kdk\int_0^t [U_1(\tau)\sin k(x-y_1(\tau)) - U_2(\tau)\cos k(x-y_1(\tau))]\, e^{-ky_2(\tau)-\lambda(t-\tau)}\left[\cos\omega(t-\tau) - \frac{\lambda}{\omega}\sin\omega(t-\tau)\right]d\tau,$$

where

$$\omega(k) = \gamma(k) - \chi(k);$$

$\gamma^2(k) = [(\rho_2-\rho_1)gk + \sigma k^3]/(\rho_1+\rho_2)$ is the dispersion relation for free waves in an ideal infinite bilayer when surface tension is taken into account;

$$\chi(k) = \frac{k\sqrt{2\gamma(k)\rho_1\rho_2\mu_1\mu_2}}{(\rho_1+\rho_2)(\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2})};\quad \lambda(k) = \chi(k) + \frac{2k^2(\rho_1\mu_1^2+\rho_2\mu_2^2)}{(\rho_1+\rho_2)(\sqrt{\rho_1\mu_1}+\sqrt{\rho_2\mu_2})^2};$$

and g is the acceleration of gravity.

The superposition of a horizontal translational motion and a vertical oscillations of the cylinder yields for the center of the cylinder a trajectory of the form

$$y_1(t) = U_0 t,\ y_2(t) = h + a\sin\Omega t. \tag{3}$$

If the amplitude of the oscillations of the cylinder are asumed to be small ($a_1 = a/h \ll 1$), the solution (1) for motion of the form (3) can be linearized in a_1, and transforming to a moving coordinate system $x_1 = U_0 t - x$, it can be written as

$$\eta = \eta_0 + a_1(\eta_c\cos\Omega t + \eta_s\sin\Omega t) + O(a_1^2),$$

where $\eta_0(x_1, t) = -U_0\int_0^\infty dk\int_0^t A(k, p)\sin k(U_0 p - x_1)\,dp;$

$$\eta_c(x_1, t) = \Omega h\int_0^\infty dk\int_0^t A(k, p)\cos\Omega p\cos k(U_0 p - x_1)\,dp; \tag{4}$$

$$\eta_s(x_1, t) = \Omega h\int_0^\infty dk\int_0^t A(k, p)\sin\Omega p\cos k(U_0 p - x_1)\,dp, \tag{5}$$

$$A(k, p) = \frac{2kR^2}{2+\varepsilon}e^{-kh-\lambda p}\left(\cos\omega p - \frac{\lambda}{\omega}\sin\omega p\right).$$

The term η_0 in the limit $t \to \infty$ describes the stationary wave motion arising from the streamlining of the cylinder by a uniform flow with velocity U_0. The asymptotic behavior of this function is studied in the limit x_1, $t \to \infty$ using the method of stationary phase. Considering only stationary points which are near the inviscid solution, and without the inclusion of surface tension (σ = 0) we obtain

$$\eta_0 \approx \frac{2\pi R^2 U_0}{2+\varepsilon}|\beta| k_0\exp[-k_0 h - \beta\lambda(k_0)x_1]\left[\sin k_0 x_1 + \frac{\lambda(k_0)}{k_0 U_0}\cos k_0 x_1\right]\quad (x_1, t\to\infty), \tag{6}$$

where $\beta^{-1} = U_0 - \frac{1}{2}\left(\frac{g_1}{k_0}\right)^{1/2} + \frac{5}{4}\alpha_1 k_0^{1/4}$; $g_1 = \varepsilon g/(2+\varepsilon)$, $\alpha_1 = \chi k^{-5/4}$; and k_0 is the root of the equation $\alpha_1 k^{3/4} + U_0 k^{1/2} - g_1^{1/2} = 0$; for small α_1 the approximate equation $k_0 \approx g_1(U_0^{-1/2} - \alpha_1 g_1^{1/4}/2U_0^2)^4$ can be used. The solution (6) compares well with the numerical calculations presented in [8].

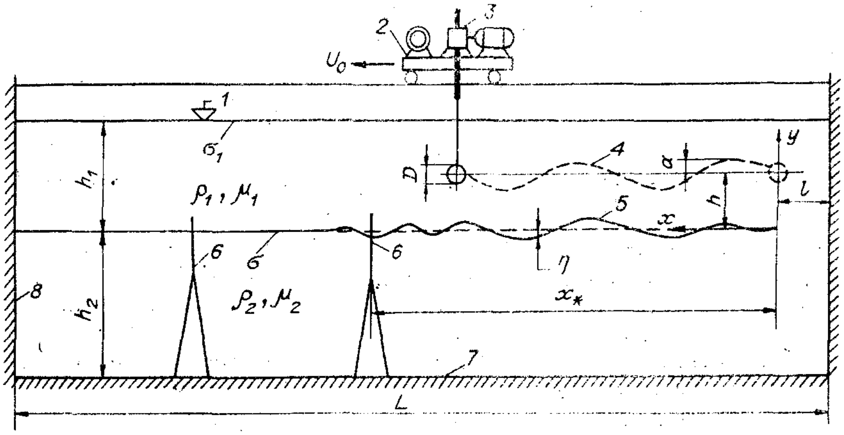

Fig. 1

The asymptotic behavior of the functions η_s and η_c are studied in a similar way. The phase functions in the integrals (4) and (5) are

$$\Psi_{1,2}(k, p) = k(U_0 p - x_1) \pm p[\Omega + \omega(k)],$$
$$\Psi_{3,4}(k, p) = k(U_0 p - x_1) \pm p[\Omega - \omega(k)].$$

The stationary points are the solutions of the system of equation

$$\partial\Psi_i/\partial k = 0, \ \partial\Psi_i/\partial p = 0 \ (i = 1, \ldots, 4).$$

The simplest case is that of an ideal fluid without surface tension ($\mu_1 = \mu_2 = \sigma = 0$). In this case the function Ψ_1 has no stationary points and the only stationary point of Ψ_2 is given by the expression

$$k_1 = \left(\frac{\sqrt{g_1 + 4U_0\Omega} + \sqrt{g_1}}{2U_0}\right)^2, \quad p_1 = \frac{x_1}{U_0}\left(1 + \sqrt{\frac{g_1}{g_1 + 4U_0\Omega}}\right)$$

and it exists only when $x_1 > 0$. The function Ψ_3 has two different stationary points when $\Omega < \Omega_* = g_1/4U_0$:

$$k_{2,3} = \left(\frac{\sqrt{g_1} \mp \sqrt{g_1 - 4U_0\Omega}}{2U_0}\right)^2, \quad p_{2,3} = \frac{x_1}{U_0}\left(1 \mp \sqrt{\frac{g_1}{g_1 - 4U_0\Omega}}\right),$$

where the point k_2, p_2 contributes only when $x_1 < 0$ and the point k_3, p_3 contributes only when $x_1 > 0$. When $\Omega = \Omega_*$ the function Ψ_3 has a multiple stationary point, which corresponds to a resonant excitation of a driven wave. When $\Omega > \Omega_*$ there are no stationary points of Ψ_3. The function Ψ_4 has a single stationary point only when $x_1 > 0$:

$$k_4 = \left(\frac{\sqrt{g_1 + 4U_0\Omega} - \sqrt{g_1}}{2U_0}\right)^2, \quad p_4 = \frac{x_1}{U_0}\left(1 - \sqrt{\frac{g_1}{g_1 + 4U_0\Omega}}\right).$$

Hence, when $\Omega > \Omega_*$ there exists two waves propagating downstream and when $\Omega < \Omega_*$ there are four waves, three of which propagate downstream and one upstream.

When surface tension is taken into account, numerical methods must be used to determine the stationary points, independently of whether the viscosity is taken into account. The total number of waves in this case can be as many as six. The presence of viscosity means that because of the exponential factor in (4) and (5), the resonant divergence of the solution at $\Omega = \Omega_*$ is removed, and the damping effect of viscosity at relatively long wavelengths is more significant for internal waves on the surface than for surface waves.

The experimental set-up is shown schematically in Fig. 1, where 1 is the free surface, 2 is a self-moving trolley, 3 is an actuator for vertical oscillations, 4 is the trajectory of the axis of the cylinder, 5 is the surface, 6 is a fixed wavemeter, 7 is the bottom of

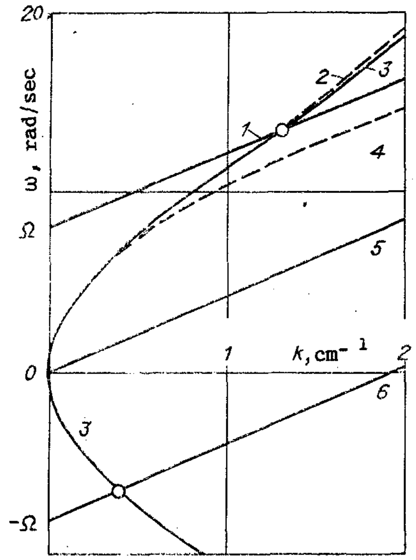

Fig. 2

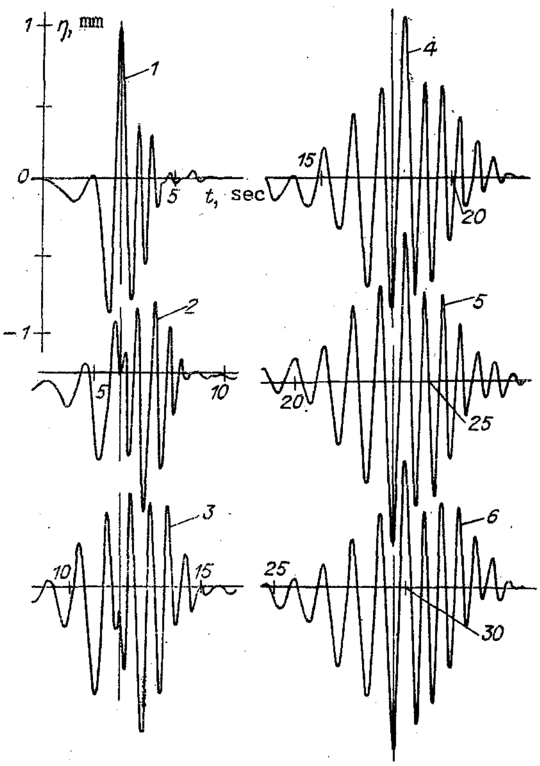

Fig. 3

the channel, and 8 is its lateral wall. The most significant parameters of the problem are also shown. Water was used as the lower layer and kerosene was used as the upper layer.

The deflection η of the surface from the equilibrium position was measured as a function of time for $0 \leq t \leq t_1$ at several fixed values of $x = x_*$. The time t_1 was chosen such that the wave reflected from the lateral wall of the channel did not as yet reach the point x_* at the instant of detection. The parameter U_0 and Ω were varied. The other quantities, upon which η depends in the general case, were held constant: $L = 4.8$ m, $\ell > 40$ cm, $h_1 = 15$ cm, $h_2 = 30$ cm, $D = 2R = 1$ cm, $a = 0.5$ cm, $h = 2$ cm, $\rho_1 = 0.8$ g/cm³, $\rho_2 = 1$ g/cm³, $\mu_1 = 0.01296$ g/(cm-sec), $\mu_2 = 0.0108$ g/(cm-sec), $\sigma = 34 \cdot 10^{-3}$ N/m, $\sigma_1 = 27 \cdot 10^{-3}$ N/m. The width of the channel was $B = 20$ cm and the gap between the lateral walls of the channel and the faces of the cylinder was 0.2 cm. Measurements at different points along the width of the channel showed that the wave was practically two-dimensional.

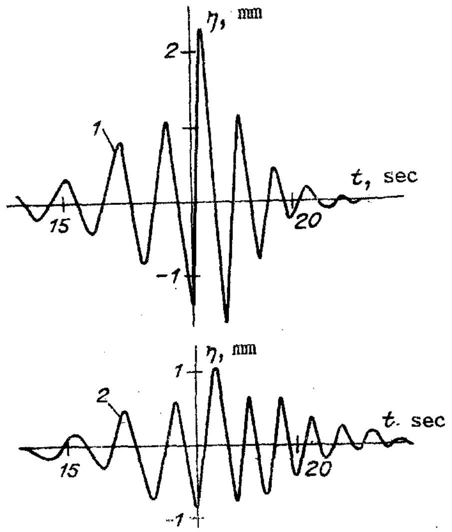

Fig. 4

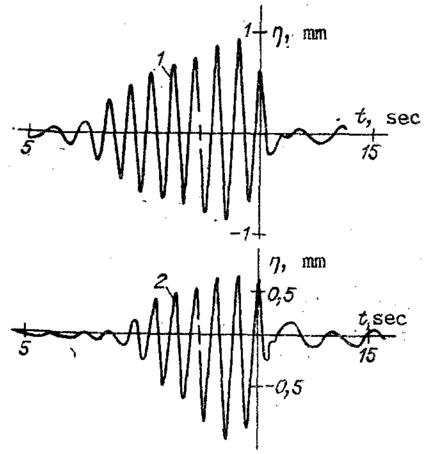

Fig. 5

The maximum value of η in the experiment was of order 1 mm, and the characteristic frequency of the oscillations did not exceed 1.5 Hz. The displacement η was transformed into an electrical signal, as in [7], by wavemeters, in which the significant difference between the electrical conductivities of the two layers is used. The transformation was linear, with a sensitivity of order 100 mV/mm, a lower threshold of about 50 μ, and uniform frequency characteristics, at least between 0 and 4 Hz. The electrical signal was detected by a two-channel recorder with suitable characteristics.

The main attention in the experiment was focused on the aforementioned critical and near-critical regimes. Here the function $\eta(t)$ was quite complicated and it was difficult to formulate a rigorous quantitative treatment of the random errors of measurement for this quantity. We note that repeated measurements under the same conditions showed that the phase diagram of the fundamental wavetrain was reproduced to within the accuracy of the scale of the recording and the largest wave amplitude varied by $\pm 10\%$.

Figure 2, curve 3 shows the dispersion dependence (2), describing the relation between the wavenumber k and the frequency ω for internal waves with the parameters given above. This dependence is well supported by earlier experiments [7]. The function $\omega(k)$ is symmetric with respect to the k axis. For comparison we show the upper branches of the dispersion curves for inviscid fluids ($\mu_{1,2} = 0$, curve 2) and for pure gravity waves ($\mu_{1,2} = 0$, $\sigma = 0$, curve 4).

In order to determine which of the possible vibrations are excited in the system by a given excitation, the dependence $\omega_*(k)$ characterizing the excitation must be plotted in the (k, ω) plane. For the problem considered here $\omega_*(k)$ is the set

$$\omega_*(k) = \{U_0 k, U_0 k + \Omega, U_0 k - \Omega\},$$

whose separate functions are labeled in Fig. 2 by the figures 5, 1, and 6. For a purely vibrational motion of the cylinder $\omega_* = \pm\Omega$, and for purely translational motion $\omega_* = \pm U_0 k$. The points of intersection of $\omega_*(k)$ and $\omega(k)$ determine the values of k and ω of the excited waves.

The critical regimes correspond to the points of tangency of $\omega_*(k)$ and $\omega(k)$. Two such regimes were realized in the experiments: regime I with $U_0 = 8.44$ cm/sec, $\Omega/2\pi = 0.51$ Hz, and regime II with $U_0 = 11.9$ cm/sec, $\Omega/2\pi = 0.36$ Hz. The quantity $\Omega U_0/g$ was 0.248 and 0.247 in these two cases, respectively, and in both cases the point of tangency lies in the region of pure gravity waves. Regime III was also realized with $U_0 = 4.28$ cm/sec, $\Omega/2\pi = 1.03$ Hz, for which $\Omega U_0/g_1 = 0.254$, such that for gravity waves this would be a critical regime. But in a system with surface tension there exist only intersection points of $\omega(k)$ and $\omega_*(k)$ in this regime. The quantity U_0 was varied in the neighborhoods of regimes I and III.

In Fig. 3 we show a succession of records of $\eta(t)$ in regime I, taken by wavelengths placed at different distances x_* (curves 1 through 6 correspond to the values $x_* = 0.25$; 0.5; 1; 1.5; 2; 2.5 m). The time is measured from the start of motion of the cylinder.

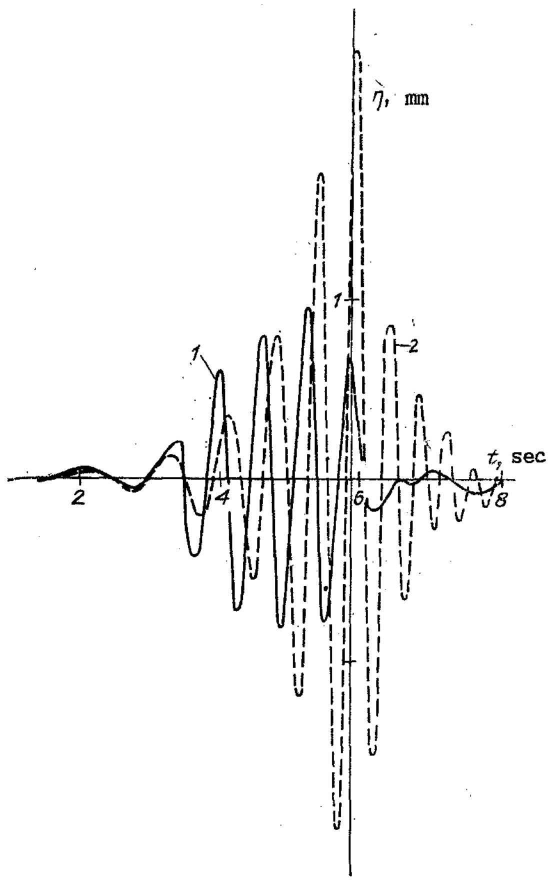

Fig. 6

The parameters U_0, Ω, and a became constant after a time not larger than 0.2 sec. The records were taken up to the instant of stopping of the cylinder and the arrival of the reflected wave at the point of measurement. The times where the axis of the cylinder passed above the wavemeters are noted by vertical lines.

We see from Fig. 3 that in the critical regime, the wave field exists both in front of and behind the cylinder and there is a sharp change in the vibration frequency of the free surface below the cylinder. In this regime the curves $\omega(k)$ and $\omega_*(k)$ have three common points: a point of tangency on the upper branch of the dispersion curve, and two intersection points; one on the lower branch of $\omega(k)$, and the other on the upper branch. Therefore three stationary waves are excited in the system with wavelengths and frequencies corresponding to these points. The only wave that is propagated in front of the cylinder is the wave corresponding to the point of tangency. Behind the cylinder there is a superposition of all three waves.

If the maximum amplitude of the wavetrain is compared with the maximum amplitudes resulting from purely translational motion with U_0 = 8.44 cm/sec or purely vibrational motion with $\Omega/2\pi$ = 0.51 Hz, then one may note the significant change not only in the phase relation, but also in the maximum amplitudes of the waves. In particular, for purely translational motion, waves were not excited in the system at all for $U_0 < 9.6$ cm/sec because of the effect of the surface tension (the curve $\omega_*(k) = U_0k$ did not intersect $\omega(k)$ in this case). For purely vibrational motion of the cylinder with $\Omega/2\pi$ = 0.51 Hz the wave amplitude did not exceed 0.03 mm.

Variation of U_0 in the neighborhood of the critical value showed that growth of the waves of the kind occurring in resonance phenomena in dynamical systems with significant damping was evident in the neighborhood of the critical regime. The phase relation of the waves also changed smoothly. This indicates that it is fundamentally necessary to take into account viscosity in the mathematical model.

Figure 4 shows a comparison of the calculated and experimental data for regime I at x_* = 1.5 m (1 is the calculation performed by a numerical integration of (1) with $y_1(t) = U_0(t)$, $y_2(t) = h - a \sin \Omega t$ and the parameters of the motion given above; 2 is the experimental curve). There is agreement in all cases except the following: 1) in the calculations it was assumed that L, $B \to \infty$; arguments for the validity of this assumption were given above; 2) the calculations were done for $h_{1,2} \to \infty$; this has not led to significant disagreement with earlier experimental data (see, for example, [7]), and also this is supported by visual observations of the free surface; 3) in the theory the dipole imitating the cylinder is turned on instantaneously, whereas in reality the time for U_0, Ω, and a to reach constant values is small but finite. This leads to an ambiguity between the calculations and the experiment of the points of the trajectory of the cylinder where it passes above the wavemeter.

The latter effect can be estimated by varying the initial phase ϕ of the vibrational motion of the cylinder between 0 and 2π in both the calculation and the experiment. In the calculated results essentially the only effect is a displacement of the point of the stationary wave train at which the vibrational frequency of η changes. Because of the fact that h was not sufficiently large in the experiment, there was also a variation of the maximum value of η in the wavetrain of about 10%. The calculation and experiment were compared by choosing experimental records and calculations in which the change of the vibrational frequency occurred at the same point.

The critical regime II was considered in order to examine the situation for longer wavelengths, when the viscous damping is weaker. Unfortunately the values of η were so small in this case that they could not be detected, even with extremely sensitive wavemeters. One can only conclude that the growth of the waves is bounded in this regime.

On the other hand, amplification of waves for a complex excitation (in comparison with that which occurs for the separate components of the wave) can be observed in noncritical conditions as well. An example is the data for regime III with $x_* = 1.5$ m as shown in Fig. 5 (1 shows the calculated result from (3); 2 shows the experimental result). In this case only waves in front of the cylinder are amplified. At this distance x_* the wavetrain becomes practically constant. Its characteristic frequency is determined by the intersection point of $\omega(k)$ and $\omega_*(k) = U_0k + \Omega$.

In Fig. 6 the calculated results are shown for regime III with $x_* = 0.25$ m, with the same conditions as in Fig. 5 and with (curve 1) and without (curve 2) the effects of viscosity and surface tension.

The data of Figs. 4 through 6 shows that the inclusion of viscosity, even in the linear theory, leads to fairly good agreement with experiment, particularly for the phase diagram of the waves. The somewhat too large wave amplitudes in the calculations are apparently due to the omission of the nonlinear terms in the boundary condition on the free surface.

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